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Mathematical Tables
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Aids to Computation

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Numerical Testing of Series by Calculations to many Places of Decimals

1. The first query (Q 1) and the first four replies (QR 1, 2, 3, 4) all dealt with the uses of tables to many places of decimals. This note indicates yet another use for many-figure calculations.

New functions are continually being studied, and new expansions or approximations are constantly being derived for these, as well as for other better-known functions. Whenever a new development is derived, it is clear that it is desirable to test its accuracy by using it *numerically* whenever an alternative process of calculation is available. Such a test is particularly desirable when the new development results from complicated processes and involves complicated coefficients.

In the case of a newly-derived series there is need for the calculation to be to many decimals in order to ensure adequate 'dispersion' of the effects of the various terms, and especially so that there may be a fair amount of variation in dispersion for different values of the variable or variables occurring in the series. Thus, in testing a series of which, say, the first six terms are available, it is often convenient to have the ratio of successive terms of the order of 500 to 2000, so that each term has about three clear digits before the effect of the next term begins; a calculation to about eighteen digits would then be needed for the six terms. Again, it is sometimes possible when testing a complicated series numerically to use even more figures so that the effect of still later terms, not obtained or easily obtainable analytically, may be studied and approximate numerical values derived for the coefficients.

2. All these points may be illustrated by a study of the new series, given by LEHMER[1] (p. 133) for the very well-known Bessel function $I_n(x)$. This important expansion, as given by Lehmer, contains a number of misprints and errors, which will now be located and removed, mainly by means of numerical checks. The writer is fortunate in having available the following results, obtained in collaboration with W. G. BICKLEY during the preparation of a second volume of Bessel functions by the Committee for the Calculation of Mathematical Tables of the B.A.A.S.

n	$I_n(3n/4)$	$\ln I_n(3n/4)$
4	0.32570 51819 37935	− 1.12176 26566 70127
8	0.42296 60682 03539	− 0.86046 33201 66327
12	0.63312 41362 66149	− 0.45708 87682 19174
16	1.00486 91224 26993	+ 0.00485 73065 90037
20	1.64701 52535 01582	+ 0.49896 47125 37913
24	2.75504 76802 56368	+ 1.01343 47492 63236

3. First, a comparison between Lehmer's value of $\psi_3(x^2)$ on p. 133 and the coefficient of ν^{-3} in Watson's formula[2] (2) indicates a discrepancy in the sign of the coefficient 16. For this early term in the series the methods suggested by both writers are easy to apply and it turns out that Lehmer's relation should read

$$5760\,\psi_3(x^2) = 375x^6 - 3654x^4 + 1512x^2 + 16$$

(The anomaly of this term +16 is curious; it is the only one to break the alternation of sign which holds elsewhere.) The corresponding term on p. 134 also needs correction:

$$for\ -1044097/175781250n^3\ read\ -36521/7031250n^3.$$

Secondly, the sign of the term in n^{-7} is + on p. 134 and − on p. 135. As it is a late and small term (0.0^{15} 45844 in Lehmer's illustration) the sign '−' will be 'adopted' and verification left to the numerical test.

4. The formula for $\ln I_n(3n/4)$ may now be used, so modified in two places from the form given on p. 134, for the values $n = 8(4)24$. The results obtained, using the *series curtailed at the term in* n^{-8}, are listed in col. (*A*); the smallest term used, that in n^{-8}, is listed in column (*B*). Calculations are to 15 working decimals, no more being available for comparison.

n	A	B $10^{-15}\times$	$C=$BAAS$-A$ $10^{-15}\times$	$D=$BAAS$-A_c$ $10^{-15}\times$	$E=D\times n^{10}$	Δ	F
4				3292 38885	0.3452		0.25
						1.2464	
8	− 0.86046 33216 70947	69 52010	15 04620	14 82320	1.5916		1.58
						1.3244	
12	− 0.45708 87682 88568	2 71257	69394	47094	2.9160		2.91
						1.328	
16	+ 0.00485 73065 63877	27156	26160	3860	4.244		4.24
						1.33	
20	+ 0.49896 47125 15069	4556	22844	544	5.57		5.57
						1.3(4)	
24	+ 1.01343 47492 40827	1060	22409	109	6.9(1)		6.90

In col. (*C*) are given the differences between BAAS results listed above and those in col. (*A*). The results in col. (*C*) for large n suggest strongly that the differences are tending to a non-zero limit, that is, that there is a (numerical) error in the constant term $-\frac{1}{2}\ln(5\pi/2)$; this turns out to be the case: on p. 135,

$$for\ -1.03051\ 03088\ 84078\ 40331,\ read\ -1.03051\ 03088\ 61777\ 61966.$$

If this term is corrected, the values in col. (*D*) remain as the differences between the BAAS results and values given by Lehmer's series, corrected, as far as the term in n^{-8}. These corrected values (not given) are denoted by (A_c). The value for $n = 4$ in col. (*D*) was computed directly from the corrected series.

5. Provided that no early term is in error, it may be expected that these differences (*D*) will be of the form

$$L/n^9 + M/n^{10} + N/n^{11} + \cdots$$

in which later terms will be less and less important as n increases. To estimate the first two coefficients L and M several methods have been tried; the most convenient for the present example seems to be multiplication by n^{10} followed by formation of differences: col. (*E*), with Δ, results. As it was hoped, for high n the first difference Δ seems to tend to a limit as the effect of higher terms N/n^{11}, etc. dies out; this limit is slightly greater than 1.33, although more figures are needed for $n = 20$ and $n = 24$ to estimate it more closely. Take $L = 1.33/4 = 0.3325$ (4 being the interval in n); then from

the value for $n = 20$, using $0.3325n + M = 5.57$, it follows that $M = -1.08$. These two coefficients lead to the approximate values in col. (F); the effect of omitted terms is obvious for $n = 4$, but is less than 1 per cent elsewhere with $8 \leq n \leq 24$. Extrapolation outside the limit $n = 24$ may be a little unsafe; but the last two terms given below for $n = 100$ are probably correct.

6. The approximate linearity of the values in col. (E) for large n also indicates the absence of gross errors in earlier terms; the '$-$' sign of the term in n^{-7} is confirmed, for instance. It is instructive to consider the values which would have resulted from a wrong sign for this term; *assuming the error in the constant term to have been eliminated*, the values in col. (D') would have been found in place of those in col. (D). Multiplication by n^{10} would then have given col. (E'), obviously not linear.

n	D' $10^{-10}\times$	$E' = D' \times n^{10}$	Δ	G	H
4	− 52669 87883	− 5.523			
			− 39.830	8.212	$(8/4)^3 = 8$
8	− 422 38202	− 45.353			
			− 110.169	3.429	$(12/8)^3 = 3.375$
12	− 25 11765	− 155.522			
			− 215.790	2.388	$(16/12)^3 = 2.370$
16	− 3 37706	− 371.312			
			− 356.63	1.960	$(20/16)^3 = 1.953$
20	− 71088	− 727.94			
			− 532.7	1.732	$(24/20)^3 = 1.728$
24	− 19883	− 1260.6			

F
0.25
1.58
2.91
4.24
5.57
6.90

To find in which term an error seems likely, col. (G) gives the ratios of successive terms in col. (E'). These values are clearly approximate *cubes* of ratios of successive values of n, thus indicating the term in n^{-7} of the original series as the almost certain source of error. If two or more errors of comparable magnitude had been present this would have been evident from a considerable variation of the power of $(n + 4)/n$ represented by the values in col. (G). In such cases there will be at least two errors to find and a general systematic search is indicated, starting at and near the term giving the power most frequently represented by col. (G).

7. For $n = 100$, the following table of terms, corrected and extended from that given by Lehmer on p. 135, results:

	15.13877 11331 89030 86048
−	2.30258 50929 94045 68402
−	1.03051 03088 61777 61966
−	6 66666 66666 66667
−	31680 00000 00000
−	51 94097 77778
+	1 77894 60480
+	1115 32887
−	32 28031
−	45844
+	1166
+	33
−	1
The sum is	11.80560 58916 51420 31925

Thus $I_{100}(75) = 134001.44891\ 20951\ 594$

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[1] D. H. LEHMER, *MTAC*, p. 133–135.
[2] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge, 1922, p. 228.

RECENT MATHEMATICAL TABLES

158[A].—H. S. UHLER, *Exact Values of the first* 200 *Factorials*, New Haven, Conn., privately published by H. S. Uhler, 1944, 24 p. + cover. 16×25.5 cm. Photo-litho print. $.80

In *MTAC*, p. 163, 125, are records of unpublished tables of $n!$ by JOFFE, $n \leqslant 100$, and by SALZER and HILLMAN, $n \leqslant 120$. Among previously printed tables the most extensive are those of PETERS and STEIN (J. T. PETERS, *Zehnstellige Logarithmentafel* . . ., v. 1, 1922, *Anhang*, p. 58), $n \leqslant 60$; of F. ROBBINS, R. So. Edinburgh, *Trans.*, v. 52, 1917, p. 167–174, $n \leqslant 50$; and of L. POTIN, *Formules et Tables Numériques* . . ., Paris, 1925, p. 836, $n \leqslant 50$ (with four errors). The Introduction of the table under review, $n \leqslant 200$, (p. [iii–vi]) includes an account of "details of computation," "additional checks," "comparison with the results of others," "peculiarities of the table."

159[A].—WERNER F. VOGEL, *Angular Spacing Tables*, Detroit, Michigan, Vinco Corporation, 1943. iv, 233 p., hinged. 21.8 × 28.5 cm. $10.00

Especially in connection with the spacing of teeth in such things as precision gears, splines, and index plates, are modern demands of a very high order. It is desirable to know within 0″.0005 how many degrees, minutes and seconds are contained in angles between a line from the center of the gear through the center of one tooth to the line from the center of the gear through the center of any subsequent tooth on the gear. The main table of this volume (p. 1–208) provides such information for gears with 2(1)200 teeth. The presentation of the material is very clear. While seconds are given to 3D, in every case where this decimal part is $\geqslant .5$ the entry appears as, for example, 35.*510, and the following footnote appears on each page: "*Star indicates that one second must be added when decimals are disregarded." For all practical purposes accuracy to the nearest 1″ suffices.

In the next three tables (p. 212–217), the values are given for $90°/n$, $180°/n$, $360°/n$ where n is the number of sides of a regular polygon, $n = 4(1)200$. In the first table the values of these angles are in degrees, minutes, and seconds to 4D; in the second table they are given in degrees to 7D; and in the third table in radians to 10D. In each of these tables values corresponding to $n = 1(1)3$ are also given. On p. 218–227 is an 8-place table for converting any number of minutes and seconds up to 1° into a decimal of a degree. The next table is for the conversion of 0(0″.001)0″.999 to decimals of a degree, 8D. Under the heading "Important Constants," π and $1/\pi$ are given to 70D; $n \cdot \pi$ and $n \cdot (1/\pi)$, $n = 1(1)9$, to 35D; π/n and $1/(n\pi)$, $n = 1(1)19$, to 35D; and $n \cdot \pi/180$ and $n \cdot 180/\pi$, $n = 1(1)9$, to 35D.

Mr. Vogel is now a member of the staff of Wayne University Engineering College in Detroit. As a former member of the Computing staff of J. T. PETERS at the Astronomisches Recheninstitut of Berlin-Lichterfelde, we may be sure that he was most meticulous in every type of check to ensure the accuracy of his published tables.

R. C. A.

160[D, P].—GENERAL ELECTRIC CO., *Trigonometric Functions of Half Center Angle of Regular Polygons* (Standard Tables Division, Design Data, Mathematical Tables, Section G 902.4), February 22, 1944. 10 p. 20.5 × 26.5 cm.

If n equals the number of sides of a regular polygon $180°/n$ is the number of degrees in half the center angle, subtended by a side. For $n = 3(1)500$ the angles $180°/n$ are expressed to the nearest 0".01. Then follow 7-place values of their natural sines, cosines, tangents and secants. Such tables are used in manufacture of machines with integral numbers of commutator bars or slots. An earlier edition of this table (1943) was of $m \sin (180°/n)$, only for $n = 3(1)500$, $m = 1, 2, 3$.

R. C. A.

161[F].—J. A. Todd, "A table of partitions," London Math. So., *Proc.*, v. 48, 1943, p. 229–240; 1944, p. 241–242.

The author denotes by $p(n, m)$ the number of partitions of n into precisely m parts, and gives a table of $p(n, m)$ for $m \leqslant (n + 1)/2$ and for $n \leqslant 100$. This is an extension of an old table of Euler[1] which gives the same function for $m \leqslant 20$ and $n \leqslant 59$.

It is easily seen that $p(n, m)$ is also the number of partitions of n in which the largest part is m. By considering separately the cases in which m occurs once and more than once as the largest part, we see at once that

$$p(n, m) = p(n - 1, m - 1) + p(n - m, m).$$

This is the recurrence relation by means of which the author has calculated his table. It is clear that if $m > n/2$ the partitions enumerated by $p(n, m)$ will involve m only once, the remaining parts giving an unrestricted partition of $n - m$; that is

$$p(n, m) = p(n - m), \qquad m > n/2,$$

where $p(k)$ is the number of unrestricted partitions of k. This fact enables the author to dispense with one half of the values of $p(n, m)$ by giving a small auxiliary table of $p(k)$ for $k \leqslant 50$.

The reviewer has compared the table with that of Euler and has found complete agreement where they overlap.

Interest in the present table is heightened by quite recent investigations of Gupta,[2] Erdös & Lehner[3] and others on the function $p(n, m)$. The author, who is apparently unaware of these results, states that $p(n, m)$ is of the "order of magnitude" $n^{m-1}/[m!(m - 1)!]$ for fixed m as $n \to \infty$. Gupta has shown much more, namely

$$\frac{1}{m!}\binom{n-1}{m-1} \leqslant p(n, m) \leqslant \frac{1}{m!}\binom{n-1+c}{m-1},$$

where c can be taken as small as $m(m - 1)/4$, and perhaps slightly less. Another important question of current interest on which this table may shed some light is that of the maximum value of $p(n, m)$ as a function of m. In other words, among all partitions of n what size for the largest part is most popular, or to put it another way, what is the most popular number of parts? For $n = 90$ the table shows a maximum at $m = 17$. In fact $p(90, 17) = 3483945$. However a very close second is $p(90, 16) = 3483604$. According to the theory the answer is approximately

$$(3n/2)^{\frac{1}{2}} \cdot [\log (n/\pi)]$$

which for $n = 90$ gives 16.64.

A companion double-entry table of restricted partitions is that of Gupta[4] which gives for $n \leqslant 300$ the number $q(n, m)$ of partitions of n with m as the smallest element or, what is the same, the number of partitions of n in which the largest part occurs precisely m times. This perhaps less natural function satisfies the recurrence formula

$$q(n, m) = q(n - m, m) + q(n + 1, m + 1)$$

and was used by GUPTA[4] (proceeding by descent on m) to construct a table of $p(n) = q(n+1, 1)$ for $n \leqslant 600$.

D. H. L.

[1] L. EULER, "De partitione numerorum," Acad. Sci. Petrop., *Novi Commentarii*, v. 3 (1750–1), 1753, p. 125–169. *Commentationes Arithmeticae*. St. Petersburg, v. 1, 1849, p. 73–101, *Opera Omnia*, s. 1, v. 2, 1915, p. 254–294. CHRYSTAL in his algebra refers to a smaller table of Euler in his *Introductio in Analysin Infinitorum* [Lausanne, 1748, chapter 16; *Opera Omnia*, s. 1, v. 8, 1922, p. 327–328] and many later writers, including the author, have followed this reference.

[2] H. GUPTA, "An inequality in partitions," Bombay Univ., *Jour.*, v. 9, (1942), p. 16–18.

[3] P. ERDÖS & J. LEHNER, "The distribution of the number of summands in the partitions of a positive integer," *Duke Math. J.*, v. 8, 1941, p. 335–345.

[4] H. GUPTA, *Tables of Partitions*, Madras, 1939.

162[I].—NYMTP, A. N. LOWAN, technical director, *Tables of Lagrangian Interpolation Coefficients*, New York, Columbia University Press, 1944, xxxvi, 392 p., 19.6 × 26.5 cm. Reproduced by a photo offset process. $5.00.

This interesting volume gives tables of the coefficients $A_k^{(n)}(p)$ of Lagrange's interpolation formula for equally spaced arguments:

$$f(a + ph) = \sum_{k=1-\nu}^{n-\nu} A_k^{(n)}(p) f(a + kh) + R_n(f)$$

where $\nu = [(n + 1)/2]$, and h is the interval of the argument for which f is tabulated. The actual formula for $A_k^{(n)}(p)$ can be written

$$A_k^{(n)}(p) = \frac{(-1)^{n+k+\nu}\Gamma(p + \nu)}{(p - k)\Gamma(p - n + \nu)\Gamma(n - \nu - k + 1)\Gamma(\nu + k)}$$

from which it is clear that $A_k^{(n)}(p)$ is a polynomial in p of degree $n - 1$. The values of these coefficients are tabulated for

$$1 - \nu \leqslant k \leqslant n - \nu$$
$$3 \leqslant n \leqslant 11$$

The range of the argument p varies with n according to the following scheme:

n	range of p
3	− 1(.0001) 1
4	− 1(.001)0(.0001)1(.001)2
5	− 2(.001)2
6	− 2(.01)0(.001)1(.01)3
7	− 3(.01) − 1(.001)1(.01)3
8	− 3(.1)0(.001)1(.1)4
9	− 4(.1)4
10	− 4(.1)5
11	− 5(.1)5

From this it is seen that of these nine tables the first two are the main ones and that for $n < 9$ the important range, between 0 and 1, is pretty finely divided.

There are two other tables of $A_k^{(n)}(p)$ intended for use in subtabulation. These are for $n \leqslant 9$ and give exact values of $A_k^{(n)}(p)$ for p at intervals of .1 and 1/12. With these tables a given numerical table may be enlarged m-fold by a single subtabulation for $m = 2, 3, 4, 5, 6, 10, 12$.

Throughout the volume all values which are not exact are given to 10 decimal places.

Finally there are two other tables for use in approximate integration. These give the polynomials

$$M_k^{(n)}(p) = \int_0^p A_k^{(n)}(t)dt$$

and the coefficients

$$M_k^{(n)}(m+1) - M_k^{(n)}(m)$$

for integers m, k, n satisfying

$$1 - \nu \leqslant m \leqslant n - \nu - 1, \qquad 1 - \nu \leqslant k \leqslant n - \nu, \qquad 3 \leqslant n \leqslant 11.$$

There is a bibliography of 8 items describing the principal previous tables of Lagrangean coefficients. There is no mention of the obscure table of LITTLE nor of the two places where parts of the volume under review were published earlier; see *MTAC*, p. 185.

The introduction is quite interesting. It contains a brief discussion of the properties of the A's and their relations to coefficients of other interpolation formulae based on differences. There is an adequate treatment of the upper bound for the remainder $R_n(f)$ and other errors in interpolation. Finally quadrature formulae based on the Lagrangean interpolation scheme are derived and discussed.

The reviewer has recently set forth reasons (*MTAC*, p. 184) why he believes that interpolation without differences by the Lagrangean method will become more popular with users of good computing machines. This splendid volume will do much to enhance such popularity. It is a "must" for the library of every mathematical laboratory.

D. H. L.

163[I].—H. E. SALZER, "Table of coefficients for inverse interpolation with advancing differences," *J. Math. Phys.*, M.I.T., v. 23, 1944, p. 75–102. 17.5 × 25.5 cm.

This table of coefficients is a companion to a previous one[1] by the author based on the more popular central difference set up. The present table will be found more fundamental in its application since advancing differences require a minimum amount of knowledge of the given function. One associates inverse interpolation with the discovery of a real root of some equation $f(x) = \text{const}$. Often $f(x)$ is not a function which has been tabulated and published. This means that a "home-made" table of just a few entries is needed. Thus one is often working at the beginning of a table.

Let $y = f(x)$ be a monotone function of x tabulated for $x = x_0, x_0 + h, x_0 + 2h, \cdots$. If y is a given value, the problem of inverse interpolation is that of finding a number $p (0 < p < 1)$ such that $f(x_0 + ph) = y$. Let $D_k = \dfrac{\Delta^k f(x_0)}{\Delta f(x_0)}$ and $m = \dfrac{f(x_0 + ph) - f(x_0)}{\Delta f(x_0)}$ so that $0 < m < 1$; then the Gregory-Newton formula gives

$$p = m - \sum_{\nu=2} \binom{p}{\nu} D_\nu. \tag{1}$$

Complete reversion of this series for p gives

$$p = m - \binom{m}{2} D_2 - \binom{m}{3} D_3 + \frac{m(m - \frac{1}{2})(m-1)}{2} D_2^2 - \binom{m}{4} D_4 + \cdots + P(m|r_1, r_2 \cdots) D_{r_1}^{\alpha_1} D_{r_2}^{\alpha_2} \cdots + \cdots. \tag{2}$$

Here the general term is said to be of order $r_1\alpha_1 + r_2\alpha_2 + \cdots$ and the coefficient $P(m)$ is a polynomial in m. The first two pages of this paper give the explicit expansion through the terms of the 8th order, only 22 terms in all. This is a slight extension of a previous result of H.T.D.[2] now confirmed except for two minor errata.[2] The purpose of the present paper is to give a table of these 22 coefficients $P(m)$. Actually the table begins with the fourth order terms since the earlier terms are easy to calculate and have been tabulated often. (See bibliography of Lagrangean interpolation coefficients *MTAC*, p. 184–5, also the previous review, RMT **162**). The remaining 19 coefficients are naturally arranged according to their order. They are given to ten decimals for $0 \leqslant m \leqslant 1$ at intervals of .001 for the 4th and 5th order terms, .01 for 6th order terms and .1 for the 7th and 8th order terms.

It may be worth noting that the reversion of (1) accomplished once for all for a general m in (2) has in the past been done numerically for a fixed m by the method of iteration. This table should be of considerable help in handling problems in which a large number of approximate roots for some complicated transcendental equations are needed, as for instance in the determination of the characteristic values of some boundary value problem.

D. H. L.

[1] "Table of coefficients for inverse interpolation with central differences," *J. Math. Phys.*, v. 22, 1943, p. 210–224.

[2] H. T. DAVIS, *Tables of the Higher Mathematical Functions*, v. 1, Bloomington, Ind., 1933, p. 80–81. Mr. Salzer notes "The formula obtained by Davis was checked and found to be entirely correct except for two obvious printing errors, namely, p. 81 line 6 for $3!^2$ read $4!^2$ and on line 7 for + 36 read − 36."

164[I, K].—R. A. FISHER & F. YATES, *Statistical Tables for Biological, Agricultural and Medical Research*, Edinburgh, Oliver and Boyd, I. first ed., 1938, viii, 91 p.; II. second ed., revised and enlarged, 1943, viii, 98 p. 21.9 × 28.1 cm. 13s. 6d.

This volume is a collection of those tables which, in the experience of the authors, have been found most frequently useful in the application of statistical methods to biology, agriculture and medicine. Since they cover a wide range of applications, the tables will be reviewed according to their subject matter, rather than in the order in which they appear in the book.

Most of the tables refer to uses either of the analysis of variance or of the theory of statistical regression. The first seven tables are reprinted, with exceptions noted below, from R. A. FISHER, *Statistical Methods for Research Workers*, Edinburgh, Oliver and Boyd, first ed. 1925; eighth ed. 1941.

Table. I. The argument P is the probability that a normal deviate, with zero mean and unit standard deviation, falls outside the limits $(-x, +x)$. For $P = 0(.01).99$, x to 6D; for $P = .0^{3}1, .0^{3}1$, to $.0^{8}1$, x to 5D.

Table II. Contains the ordinates of the standard normal distribution to 4D, for the deviate $x = 0(.01)3(.1)3.9$. Tables III to VII were developed originally for the common tests of significance associated with the analysis of variance, namely the t, χ^2, and z tests and the test of a product-moment correlation coefficient r.

Table III. For "Student's" t distribution with n degrees of freedom, RMT 111, t is given (to 3D) such that the probability is P of a random variate falling outside the limits $(-t, +t)$. $P = .9(-.1).1, .05, .02, .01, .001$; $n = 1(1)30, 40, 60, 120, \infty$. A table of t to 5D is described in RMT 111.

For a description of Table IV, significance levels of the χ^2 distribution, see RMT 101, and for Table V, significance levels of the variance-ratio and z distributions, see RMT 99. The variance-ratio tables and the table of z for $P = .2$ do not appear in FISHER, loc. cit.

Table VI. In a bivariate normal distribution where the population correlation coefficient ρ is zero, the distribution of the sample product-moment correlation coefficient r is related to that of "Student's" t by the equation $t = r\sqrt{n}/\sqrt{1-r^2}$ where t has n degrees of freedom and $(n + 2)$ is the number of pairs of values from which r is calculated. In Table VI, P is the probability that a sample correlation coefficient falls outside the limits $(-r, +r)$. $P = .1, .05, .02, .01, .001$; $n = 1(1)20(5)50(10)100$; r to 4D throughout most of the table.

Table VII. Contains $r = \tanh z$, for $z = 0(.01)3(.1)4.9$; r to 4D for z less than 1.8, to 5D thereafter. This transformation is useful for tests of significance involving r in cases where ρ is not zero; for example, a test of the hypothesis that two populations have the same correlation coefficient. When ρ is near ± 1, the distribution of r is markedly skew even in moderately large samples; z, on the other hand, is approximately normally distributed with mean $\tanh^{-1}\rho$ and variance $1/(n-1)$ where n is as before two less than the number of pairs.

The techniques and tests associated with the analysis of variance are based on the assumption that the data follow the normal distribution. In the development of statistical techniques for problems where this assumption is far from true, one approach is to transform the data to a scale in which analysis of variance techniques can be applied, at least as a first approximation to which further refinements may be added if required. For the research worker who is not an expert statistician this device has obvious advantages, since a few techniques can be used repeatedly. We consider now three groups of tables which facilitate applications of this type.

In certain toxicological experiments it is considered that the number of subjects which succumb to a given dose (but survive any lesser dose) follows a normal distribution when plotted against a suitable function of the dose, usually the log. Thus in a group of subjects the fraction P which is killed by a dose is given theoretically by the relation

$$P = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{d} e^{-(t-m)^2/2\sigma^2}dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}dt.$$

where $x = (d - m)/\sigma$ and d is the log dose. Where this theory holds, the observed fraction P in any experiment, when transformed to the normal deviate x, shows a linear relation with log dose, the slope of the line providing an estimate of $1/\sigma$. From estimates of m and σ the dose required to kill any desired fraction and the relative performance of two poisons can be predicted. In Table IX, P (in percents) = 0.1(0.1)98.0(0.01)99.99; the normal deviate x is given to 4D with the addition of a constant value 5 in order to avoid negative deviations. The table is reprinted from C. I. Bliss, *Ann. Appl. Biol.*, v. 22, 1935, p. 138–140, with the inclusion of first differences. Apart from the constant 5 and from the fact that P refers to only a single tail, this table is simply a more extensive form of Table I. Table X, for the same purpose, gives $(-x)$ to 4D for all proper fractions $P(\leqslant \frac{1}{2})$ with denominators up to 30. There appears to be an error in the footnote to Table IX; the words "exceeded by" should be replaced by "which exceeds."

Table XI applies to later stages in the calculations. For $Y = (x + 5) = 5.0(0.1)8.9$, the functions $Y + Q/Z$ (4D), $1/Z$ (mostly 5S) and Z^2/PQ (5D) are given; $Q = 1 - P$ and Z is the ordinate of the normal curve at the point P. The function Z^2/PQ supplies the weights for fitting the line while the other two functions appear in further refinements.

Table XII gives $\sin^{-1}\sqrt{p} = \varphi$ expressed in degrees to 1D, for p (in percents) = 0(1)99 and Table XIII gives the same function for all proper fractions $(\leqslant \frac{1}{2})$ with denominators up to 30. If it is desired to apply the analysis of variance to data expressed as fractions or percentages, where the errors follow the binomial distribution, the angular transformation provides a scale in which the error variances are more nearly equal at all points. The analysis of variance in angles may be regarded as a first approximation to the maximum likelihood analysis and is often adequate. For more precise work the maximum likelihood solution itself may be obtained by means of adjustments to the angles given in Table XIV. The relevant functions are $\varphi + (90/\pi)\cot\varphi$ and $(180/\pi)\csc 2\varphi$, each given to 1D for $\varphi = 45°(1°)89°$.

Table XX supplies a transformation for the analysis of variance of *ranked* data, that is, data in which only the order 1, 2, 3, $\cdots$ of preference or performance is recorded. The table shows to 2D the mean value n_r of the r^{th} largest of a sample of n from a normal distribution with mean zero and unit standard deviation: $n = 2(1)50$; $r \leqslant \frac{1}{2}n$. Values for $r > \frac{1}{2}n$ are obtained by symmetry. The function involved is

$$n_r = \int_0^1 \frac{n!}{(r-1)!(n-r)!}p^{n-r}q^{r-1}x\,dp$$

where p is the probability that a normal deviate is less than x. The sums of squares of the mean deviations are given for $n = 2(1)50$ in Table XXI. The assumption involved in the use of Table XX is that if the ranked scores could be replaced by exact numerical scores, the latter would follow a normal distribution. When this assumption is valid, estimates made in the transformed scale are more efficient than those made in the original scale.

Many methods have been developed for fitting a polynomial regression in x when x takes the values 1, 2, $\cdots$, n. Table XXIII, which contains the orthogonal polynomials $\xi_r'(x)$, for $n = 3(1)52$, $r = 1(1)5$, is the basis of a procedure that is convenient and expeditious when a multiplying machine is available. Compare RMT 129. If $\xi_0(x) = 1$, $\xi_1(x) = x - \bar{x}$, $\xi_{r+1}(x) = \xi_1(x)\xi_r(x) - \dfrac{r^2(n^2 - r^2)}{4(4r^2 - 1)}\xi_{r-1}(x)$, the functions $\xi_r(x)$ satisfy the orthogonal relation

$$\sum_{x=1}^{n} \xi_i(x)\,\xi_j(x) = 0$$

for $i \neq j$, any fixed n. The tabulated values ξ_r' are those of the smallest multiples of the ξ_r that take integral values for every x. For $n \leqslant 8$, all individual values of $\xi_r'(x)$ are shown; for $n > 8$, only those for which $x \geqslant \frac{1}{2}n$. The multiplier and the sum of squares over $x = 1, 2, \cdots, n$ are given for each n, r. With the aid of these tables the fitting of a polynomial up to degree 5 requires little more than the calculation of the sums of products of the observations with the corresponding tabular $\xi_i'(x)$ values.

Table XXIV gives some formulae for the calculation of integrals from ordinates at equal intervals: (i) the coefficients of the Newton-Cotes formulae for 2, 4 and 6 panels with and without the use of an external ordinate: (ii) corresponding formulae expressed in terms of central differences up to the 10th, for 2(2)10 panels: (iii) the first 16 coefficients in Gregory's formula.

Table VIII, reprinted from F. YATES, R. Statist. So., *J. Suppl.*, v. 1, 1934, p. 228, is intended to provide an exact test of significance for the hypothesis that two observed fractions a/b, c/d are estimates of the same true fraction; in other words, the test of independence in a 2×2 contingency table. The customary χ^2 test, though improved by the correction for continuity, does not approximate the true significance levels very satisfactorily when $a, \cdots, d$ are small. The exact test necessitates a separate calculation for every set of values of $a, \cdots, d$. However, for all 2×2 contingency tables which have the same smallest expectation m, and the same ratio p of the smallest expectation to the smallest marginal total, the exact significance levels of χ_c (corrected for continuity) lie between two limits which can be determined. These limits are tabulated (to 2D) for $m = 1(1)\ 6, 8, 12, 24, 48, 96$; $p = 0, 0.25, 0.5$, at the 0.025 and 0.005 significance levels of χ_c, for each tail of the distribution.

Table XXII gives $\Delta^r 0^s/r!$ for $s = 2(1)25$; $r = 2(1)s$. This quantity occurs in the solution of a number of statistical problems. See *MTAC*, p. 330.

The basic plans for some common types of experimental arrangements may be constructed from Tables XV–XIX. Table XV gives a catalogue of the Latin squares from the 4×4 to the 6×6 (every standard square for the 4×4 and 5×5 and one square of each transformation set for the 6×6) and examples of Latin squares from the 7×7 to the 12×12. Table XVI contains complete sets of orthogonal Latin squares from the 3×3 to the 9×9, excluding the 6×6 for which no set exists. A number of *balanced incomplete block* arrangements are given in Table XVII. These are arrangements of v different letters (representing experimental treatments) in groups of k ($k < v$) such that any two letters appear together in a group the same number of times (λ): an example, for $v = 4$, $k = 2$, $\lambda = 1$, is the set of groups ab, ac, ad, bc, bd, cd. Because of its symmetry, such an arrangement enables differences among the groups to be eliminated from the experimental errors of the treatment comparisons and is useful, for example, when the number of treatments exceeds the number that can be compared at any one time or place. Tables XVIII and XIX constitute indexes to these arrangements, by number of replications (r) and number of units per group (k) respectively, for all cases ($r \leqslant 10$) in which the existence of an arrangement has not been disproved. For arrangements known by the authors to exist, references are given either to Table XVII or to simple methods of construction.

In the layout of replicated experiments and in the process of taking samples, there is often occasion to use some randomizing device, such as a set of numbered beans. More convenient for many purposes is a set of random digits, of which 15,000 are given in Table XXXIII.

In addition, the volume contains Tables (XXV–XXXII) of some of the standard mathematical functions: logs (to 5D); lns (to 5D); n^2 for $n = 0(1)999$; n^3, to 5S; reciprocals of 1.00 (.01)9.99, to 6D; factorials of 1(1)300(to 6S) and log $n!$, to 7D; sines, to 5D, and tangents, mostly to 5D. The table of square roots is very convenient, with $\sqrt{n}$ and $\sqrt{10n}$ placed one above the other. The tabulation of $(90° - \theta)$ tan θ instead of tan θ for θ above 60° may also be noted.

Notes on the use of the tables occupy the first 22 pages. For some Tables, e.g. XX and XXIV, the notes might profitably be amplified.

The tables are admirably printed on large pages with open-flat binding and are comfortable to the eye. As the early issue of a second edition has indicated, this volume is a most useful addition to the statistical library of the research worker. For the few errors in the tables, see MTE **47**.

In the second edition four tables have been added. Tables V_1 and V_2 give a test of significance of the difference between the means of two samples, for the case where the variances in the two populations are not assumed to be equal. The development of a test of this kind has encountered interesting problems and raised some controversy. On the classical theory of probability, the simplest approach might be to work the frequency distribution of $d = (\bar{x}_1 - \bar{x}_2)/\sqrt{s_1^2 + s_2^2}$, where s_1^2, s_2^2 are the sample estimates of the variances of $\bar{x}_1, \bar{x}_2$ respectively, for populations whose variances σ_1^2, σ_2^2 have fixed though unknown values. It has been known for some time, however, that this distribution depends on the unknown ratio σ_1^2/σ_2^2 and consequently cannot supply an exact test of significance. Another method, H. SCHEFFÉ, *Ann. Math. Statist.* v. 14, 1943, p. 35–44, is to select a different test criterion whose distribution is that of "Student's" t. This technique involves the use of an estimate of the variance of $(\bar{x}_1 - \bar{x}_2)$ which is generally less precise than $(s_1^2 + s_2^2)$. Fisher's solution, proposed first by W. V. BEHRENS, *Landwirtschaftliche Jahrbücher* v. 68, 1929, p. 807–837, is to regard the ratio σ_1^2/σ_2^2 as following its fiducial distribution while s_1^2/s_2^2 remains fixed. In effect, the "nuisance" parameter σ_1^2/σ_2^2 is integrated out of the classical solution.

For practical purposes the important question is whether the mathematical model describes the conditions under which the test is applied. If, for example, the test were used repeatedly for the comparison of experimental results on a certain crop, it would be taken for granted that σ_1^2/σ_2^2 varies from one application to another, though probably the complete fiducial distribution would not be followed since there is a positive lower limit to the error variances in field experiments. In these circumstances, over a long series of applications the significance levels might be reasonably correct. In Table V_1, due to P. V. SUKHATME, n_1, n_2-numbers of degrees of freedom in s_1^2, s_2^2 respectively = 6, 8, 12, 24, ∞; $\theta = \tan^{-1}(s_1/s_2) = 0°(15°)90°$; P = significance level = .05, .01; d to 3D. Table V_2 applies to the case where n_1 is infinite; $n_2 = 10, 12, 15, 20, 30, 60, \infty$; $\theta = 0°(10°)90°$; $P = .10, .05, .02, .01, .005, .002$; d to 3D. The authors state that from comparison with Table V_2, the values in Table V_1 may be too high by about .001.

When a series of N independent trials of an event shows a successes, Table VIII$_1$ (W. L. Stevens) provides upper and lower fiducial limits to the expected number of successes for significance levels $P = 0.1, 0.025, 0.005$. The tabulated values are Np (mostly 2D) where for an upper level

$$P = \sum_{r=0}^{a} \frac{N!}{r!(N-r)!} p^r q^{N-r}$$

$a = 0(1)14$, $a/N = .5(-.1)0$, except for $a < 4$ where more values of a/N are given. For $a \geqq 15$, $p = .5(-.1)0$, corrections are given to the estimates derived from the normal approximation to the binomial. The authors suggest (p. 25) that these corrections are not

quite exact for $a > 15$. Fiducial limits of the Poisson distribution are obtained from the value for $a/N = 0$.

In the dilution method for estimating the number of certain organisms in a solution, a series of s concentrations is made, each a times as dilute as the previous one. For each concentration n tubes are prepared and the presence or absence of the organism is noted for every tube. If λ is the true number of organisms per tube at the highest concentration, the number in the r^{th} dilution is λ/a^r. The theory of the method is that for this level the numbers of tubes having 0, 1, 2, $\cdots$ organisms are distributed in a Poisson series with mean λ/a^r. Hence the probability of a fertile tube is $(1 - e^{-\lambda/a^r})$ and the expected number of fertile tubes is

$$X = ns - n(e^{-\lambda} + e^{-\lambda/a} + \cdots + e^{-\lambda/a^{s-1}}).$$

If X is taken as the *observed* number of fertile tubes, this equation supplies an estimate of λ which, though not fully efficient, discards only about 12 per cent of the information. Solutions of the equation are provided in Table VIII$_2$ for $a = 2, 4, 10$. With two-fold dilution, $s = 4(1)10$, plus a single set of solutions for $s > 10$; $x = X/n = .4(.2)2(.5)s - 2(.2)s - .4$. For $a = 4$, $s = 4, 5, \geqq 6$ and for $a = 10$, $s \geqq 3$. The function tabulated (3D) is a quantity k such that

$$\log \lambda = x \log a - k.$$

Table XVII (Balanced incomplete blocks) has been revised so as to include four solutions discovered since the appearance of the first edition.

Explanatory notes have been added for the new tables and some revisions made in the previous notes.

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165[L].—DAVID MOSKOVITZ, "The numerical solution of Laplace's and Poisson's equations," *Quart. Appl. Math.*, v. 2, 1944, p. 148–163. 17.8 × 25.3 cm.

Using the approximate method of difference equations, based on a lattice of mesh-width h, the author has solved the two-dimensional Laplace and Poisson equations for a rectangle: $0 \leqq x \leqq nh$, $0 \leqq y \leqq mh$; the values of the solution u on the boundary are assumed given. General formulae are given in the text and tables are provided from which the solutions can be quickly computed in the cases $m = 2$, 3, or 4, n being an arbitrary integer. To use the tables, in the case of the Laplace equation, one substitutes the given boundary values in the left hand column, multiplies each entry there by the corresponding number in the proper n column, and then adds the results. In this way the second table yields $u_1(2) = u(2h, h)$ in the case $m = 2$. Using this new value, one now has a new boundary value problem for a smaller rectangle, for which the process can be repeated. In this way the function is computed at every other point. Finally, by the fundamental interpolation formula, one finds the values at the points skipped over. Similar procedures are used for the other cases. For the Poisson equation the values of the right hand member are weighted by multipliers and added in similarly. In the text there are instructions for modification of the procedure for a rectangle: $0 \leqq x \leqq nh$, $0 \leqq y \leqq (m + r)h$, where n and m are integers, but $0 < r < 1$. The entries of the tables are rounded off to 4D from calculations carried out to a larger number of decimal places.

The chief disadvantage of the results is that in the tables m is restricted to the values 2, 3 and 4. However, as the author remarks, one can use a $4 \times n$ lattice as a first approximation, after which relaxation methods can be applied to a finer mesh.

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166[L].—MITIYASU TAKAGI, "On a statistical theory of ferromagnetic crystals. Part I. Magnetization and magnetostriction," Tôhoku Imp. Univ., *Science Reports*, s. 1, v. 28, 1939, p. 20–84. 18.7 × 26.5 cm.

The appendix to this paper, p. 73–84, contains an important table of the Bessel function $I_n(x)$, together with a short description of the interesting method used for computation. The tables include $I_n(x)$ to 12S[1] for $x = 1(1)6$, $n = 0(1)11$, taken from GRAY & MATHEWS;[2] these values came in the first place from the BAASMTC report for 1889. The remaining tables are original and are to 10S for $n = 0(1)N$, with a 'tail' of 9S to 5S values for n up to N', where N and N' are as follows:

x	7	8	9	10	20	30	40	50	60	70	80	90	100
N	23	23	25	25	29	34	39	42	44	47	49	53	54
N'	25	26	27	28	34	39	43	47	50	53	57	61	64

Except for a few isolated entries, these tables cover a new range in published values; the writer, however, has access to Ms. tables[3] calculated during preparation of a second volume of tables of Bessel functions by the BAASMTC. Values common to both tables (that is all Takagi's values for $x = 7(1)10$ and for $x = 20$ up to $n = 30$) have been compared; the comparison shows his values to be correct except that, as he expected, the last figure is not reliable. In the 10-figure part of the table this last figure is in error by maximum amounts of 1 or 2 units when the leading digit is small, and up to 6 or 7 units when the leading digit is an 8 or 9, that is, the error tends to be a proportionate error of maximum amount about 7×10^{-10}. In the tails the errors are greater, up to 2 digits being affected.

It is of interest to refer back to MTE **43**, p. 200, in which J. W. WRENCH, Jr. discusses two discrepant values of $I_0(30)$, giving his own result

$$I_0(30) = 78\ 16722\ 97823{\cdot}97748\ 972$$

Takagi gives

$$= 78\ 16722\ 97900$$

whilst the BAASMTC value (obtained by W. G. BICKLEY), retaining guard figures, is

$$78\ 16722\ 97823{\cdot}97749\ 00.$$

J. C. P. MILLER

[1] The value of $I_{11}(3)$ has 11*S* only in each table.
[2] A. GRAY and G. B. MATHEWS, *A Treatise on Bessel Functions*, London, Macmillan, 1895, p. 285. Second edition, prepared by A. GRAY and T. M. MACROBERT, 1922, p. 309.
[3] Described in *MTAC*, p. 283.

167[M].—1. DAVID BIERENS DE HAAN (1822–1895), *Nouvelles Tables d'Intégrales Définies*, Edition of 1867 corrected with an English translation of the Introduction by J. F. RITT. Off-set print: New York, G. E. Stechert & Co., 1939. xviii, 716 p. 21.5 × 27 cm. $15.00.

2. CHRISTIAN FREDRIK LINDMAN (1816–1901), *Examen des Nouvelles Tables d'Intégrales Définies de M. Bierens de Haan, Amsterdam, [sic] 1867*. (K. Svenska Vetenskaps Akad., *Handlingar*, v. 24, no. 5, Stockholm, 1891.) Off-set print: New York, G. E. Stechert & Co., 1944, 231 p. 21.5 × 26.8 cm. $7.50.

Apart from having achieved fame as the outstanding mathematical bibliographer and historian of the Netherlands, Bierens de Haan put all mathematicians in his debt as the result of enormous labor in assembling and tabulating material regarding definite integrals and their values. His first great collection of about 7300 formulae was published in 1858 and filled v. 4 of the *Verhandelingen*, of the Royal Academy of Sciences of Amsterdam. But Bierens de Haan continued active research in the field and by 1862 had filled the 702 p. of v. 8 of the *Verhandelingen*, with an *Exposé de la Théorie des Propriétés, des Formules*

de Transformation, et des Méthodes d'Évaluation des Intégrales Définies. A few years later, having accumulated considerably more material he finally published at Leiden in 1867, with the aid of a royal subsidy, the volume of *Nouvelles Tables*, containing 8359 formulae, of which only about 4200 were in the volume of 1858. This earlier collection was still valuable, however, not only because of some 3000 integrals not used in the new edition, but also on account of numerous literature references omitted in the later condensation. The thousands of definite integrals and their values were almost wholly arranged in the new work in 486 "tables," mainly grouped in accordance with the functions integrated, or with the limits of integration. In the original work there were "Additions," p. 699–727 and "Corrections," p. 729–733.

Up to 1939 there was always a chance each year to pick up copies of this 1867 edition for $15.00 a copy, or less, although other copies were frequently priced much higher. With war's disruption of international second-hand book markets came the American edition. Since this edition has only 716 p. in the main body of the work, while the original edition had (as we have seen) 733 p., anyone who had not seen both editions might incorrectly infer that there was less material in the American edition. But this is not the case. All that is on p. 699–727 (with many blank pages) of the 1867 edition, is on p. 699–716 of the American edition; nearly 300 errors listed in 1867 on five pages were abolished in 1939, because all the errors in question were then corrected in the body of the text. Unfortunately these were the only errors corrected in the preparation of this edition although many others had been pointed out nearly 50 years previously.

Already in 1884 Lindman, a gymnasium mathematics teacher, had published in the *Bihang* of the Swedish Academy of Sciences, v. 10, no. 3, "Observations sur les tables d'intégrales définies de M. Bierens de Haan" (Amsterdam, 1858), 268 p. Then seven years later, in the *Handlingar*, appeared his listing of the errors in the 1867 tables. This is the work which has now been reproduced and made available for other libraries than the few which happened to possess the volume of the Swedish serial in question.

Even if it is assumed that most of the errors of the 1867 collection of tables are eliminated by the Lindman publication, the work of nearly three quarters of a century ago is wholly inadequate for modern research needs. It has been estimated that an up-to-date comprehensive table of the definite integrals now available would probably fill three or four large volumes each containing 600 to 700 pages. It was mainly with a view to providing for publication of such a work by HARRY BATEMAN that a revolving fund was secured for our Committee. All scientific workers must applaud such a magnificent undertaking, and hope for its completion in the not too distant future.

R. C. A.

168[M].—H. M. TERRILL & LUCILE SWEENY, "An extension of Dawson's table of the integral of e^{x^3}," Franklin Institute, *J.*, v. 237, 1944, p. 495–497.

This contribution from the Biochemical Research Foundation at Newark, Delaware, is an extension of a table of $I = \int_0^x e^{x^3} dx$, for the range $x = [.01(.01)1.99; 6D]$ and $x = [2; 5D]$, by H. G. DAWSON in London Math. So., *Proc.*, v. 29, 1898, p. 521–522. The extension is for the range $x = [2(.01)4; 7\text{–}9S]$.

In preparing this extension the method followed consisted in first computing, from $x = 1.9$, by direct methods, the values of the function for 22 key arguments, differing by .1, and then filling in the values for the intermediate arguments by Gregory's formula. For the direct computation two series were employed checking against each other, the first being that of Dawson, and the second the uniformly convergent series

$$I = x + x^3/3 + x^5/(5.2!) + x^7/(7.3!) + \cdots.$$

"Since this series is the same as the series for the probability integral, save for the signs of

alternate terms, only half the terms are needed. The terms of odd rank are computed, and from twice the sum of these the appropriate value of the probability integral is subtracted."

The computation was started at $x = 1.9$ to enable a check to be made with some of Dawson's results; for $x = 1.92$, the authors found that $I = 12.70733$, instead of Dawson's value 12.703175. For other Dawson errors see MTE **46**. The values of the key arguments were computed $\geqslant$ 10S, and to 7D for the intermediate ones, except at the extreme end of the table.

The integral I appears in problems of mathematical physics, hydrodynamics, etc. H.B. has supplied the following references (the authors of the table under review make no acknowledgment that information concerning references 4–5 below was supplied to them):

1. K. Terazawa, R. So. London, *Proc.*, v. 92A, 1916, p. 68; $g(x) = e^{-x^2}\int_0^x e^{t^2}dt$, as well as its first and second derivatives, are tabulated, $x = [0(.1)1(1)3, 5, 10; 3\text{–}5D]$, $x^2 = 2, 3, 5, 7$.
2. K. Terazawa, Tôhoku Univ., *Sci. Reports*, v. 6, 1917, p. 172–173; tables of $g(x)$ and first four derivatives $x = [0(.1)1, 4, 5, 10; 3\text{–}5D]$, $x^2/2 = .6(.2)6$. These functions arise in consideration of the oscillation of a deep-sea surface caused by a local disturbance.
3. E. E. Watson, *Phil. Mag.*, s. 7, v. 3, 1927, p. 850, graph of $F(R) = \int_1^R dx(\ln x)^{\frac{1}{2}} = 2\int_0^{(\ln R)^{\frac{1}{2}}} e^{t^2}dt$, $0 < R < 7$; in discussing the dispersion of an electron beam.
4. N. Kapzov & S. Gwosdower, *Z. f. Physik*, v. 45, 1927, p. 133, the function $x \cdot e^{-x^2}\int_0^x e^{t^2}dt$ is tabulated for $x = [.1, .5, .8(.2)1.2(.05)\ 2.2; 5D]$; Δ, 1.5–2.2. The use for such a table arises in discussion of oscillations in electron tubes.
5. S. Sakomoto, Sächs. Akad. d. Wissen., *Berichte ü. d. Verh., math.-phys. Kl.*, v. 80, 1928, p. 217–223, who gives tables of $g(x)$ and $g(x)/x$ for $x = [0(.01)10; 4D]$; they were needed in a problem of heat conduction under certain boundary conditions.
6. W. L. Miller & A. R. Gordon, *J. Phys. Chem.*, v. 35, 1931, p. 2878–2882, $g(x)$ for $x = 0(.01)4(.05)7.5(.1)10(.2)12$; 0–1.99; 6D, 2–4.95; 8D, 5–12; 9D. The computation for $x = 0 - 2$ was based on the uncorrected Dawson table.

In the Library of Brown University the authors have deposited a Ms. giving the values of I as follows:

$$x = [0(.01).1(.1)1.8; 10D], [1.9(.1)4; 5\text{–}9D].$$

R. C. A.

MATHEMATICAL TABLES—ERRATA

References have been made to errata in no. 7 of *MTAC*, part II, Bibliography, under Airey 19, Airy 4, 5, Bessel, Bourget, BAASMTC 1, Carrington, Colwell & Hardy, Dale, Davis & Kirkham, Dinnik 8, 9, 10, 11, 14, Doodson, Gray & Mathews, Hayashi, Jahnke & Emde (also under Bisacre), Kalähne, Karas, Lehmer, Lommel 2, 3, Maclean, Meissel 1, 5, NYMTP 3, Nicholson 1, Rayleigh 8, A. Russell (under Maclean), Schleicher, Schulze, B. A. Smith 1, Steiner, Tölke, Watson, A. G. Webster, Willson & Peirce. See also RMT **163** (Davis), **164** (Fisher & Yates), **166** (Takagi), **167** (Bierens de Haan), **168** (Dawson); UMT **24** (Potin); MAC **11** (Adams); N **21** (Degen, Wrinch).

44. BAASMTC, *Mathematical Tables*, v. 1, London, 1931.

P. 5, cos 26.1, *for* .56756 ···, *read* .56755 ···
P. 7, sin 47.6, *for* .46832 ···, *read* .45832 ···
It will be noted that in both cases a 5 has been converted to a 6.

L. J. C.

45. BROWN & SHARPE, *Formulas in Gearing with Practical Suggestions.* Seventeenth ed., second printing, Providence, R. I., 1943. Compare RMT **123**, p. 143.

There are two omissions on p. 225,

$$0.6944 = 25/36 \text{ and } 0.7556 = 34/45.$$

L. J. C.

46. H. G. DAWSON, "On the numerical value of $\int_0^h e^{x^2}dx$," London Math. So., *Proc.*, v. 29, 1898, p. 519–522. This is a table $h = [0(.01)1.99;\ 6D]$ and $h = [2;\ 5D]$. Dawson writes "The foregoing table is true to six figures." Compare RMT **168.**

1. A more precise statement of this table's accuracy, based upon differencing tests (as well as spot checking at $h = 1$ and $h = 2$) is the following: For $h \leqslant .79$, the table is good to within 2 units in the 6th decimal; for $h > .79$, the table is good to within a unit in the 6th significant figure except for a major error at $h = 1.92$ where Dawson gives 12.70*3175* which should read 12.70*7331* (obtained by the author both from Dawson's formula and from the Maclaurin series). [This result has been already published by TERRILL and SWEENY, see RMT **168.**] In particular, for $h = 1.85$ to 1.99, the last digit is meaningless.

The following corrections apply to the text: On p. 519, in the second equation, the second term in parentheses on the right side reads $e^{-(9\pi^2/4a^2)}\cdot\cos(3\pi x/a)$ and it should read $e^{-[9\pi^2/(4a^2)]}\cdot\cos(3\pi x/a)$. The last equation on p. 519 is not obtained by putting $a = 1$, but by setting $a = \frac{1}{2}$ and then subtracting the second equation from the first equation. On p. 522, the quantity $\Delta^2 u_h$, which occurs in a checking equation, is to be understood as meaning the second *central* difference and not the *advancing* difference, so that according to present day notation, it should read $\delta^2 u_h$.

H. E. SALZER

NYMTP

2. We recomputed the whole of Dawson's table using the same method as before. When the tables were compared we found 27 errors involving the last two or more places, and 47 last place errors, not including last place errors of one unit. It is apparent that any list of corrections would be rather cumbersome, and it seemed better to publish Dawson's table in a corrected edition, which appeared in Franklin Institute, *J.*, v. 238, 1944, p. 220–222.

H. M. TERRILL

47. R. A. FISHER and F. YATES, *Statistical Tables* . . . , 1938; compare MTE **9** and Corrigenda, p. 171.

P. 8, *for* $\frac{1}{8PQ} = \frac{1}{2(1 - R^2)}$, *read* $\frac{1}{2PQ} = \frac{2}{1 - R^2}$.

P. 27, $n = 3$, $P = .01$, *for* 11.341, *read* 11.345.

P. 41, 6.9, *for* 15.340, *read* 15.240; 7.1, *for* 7.5056, *read* 7.5062; 8.7, *for* 2147, *read* 2354.

P. 42, $p = 72\%$, *for* 58.7, *read* 58.1.

P. 55, $n = 24$, λ of ξ_1', *for* 1, *read* 2.

P. 88, middle *C* of piano, *for* 522, *read* 261.

P. 90, *for* 1 sack = 3 bushels, *read* 1 sack = 4 bushels.

FISHER and YATES, second ed., 1943, p. viii; compare RMT **164.**

In the second edition are the following errors:

P. 35, unit errors in last decimal places, $n_1 = 2$, $n_2 = 12$; $n_1 = 5$, $n_2 = 13$; $n_1 = 8$, $n_2 = 23$; $n_1 = 1$, $n_2 = 26$; $n_1 = 6$, $n_2 = 28$; $n_1 = 5$, $n_2 = 29$; $n_1 = 4$, $n_2 = 60$; $n_1 = 6$, $n_2 = \infty$.

P. 37, similarly 2-unit error for $n_1 = 8$, $n_2 = 5$; and unit errors as follows: $n_1 = 8$, $n_2 = 1$; $n_1 = 1$, $n_2 = 2$; $n_1 = 8$, $n_2 = 2$; $n_1 = 2$, $n_2 = 3$; $n_1 = 2$, $n_2 = 11$; $n_1 = 1$, $n_2 = \infty$.

These slips were discovered by comparison with the tables reviewed in RMT **102**.

R. C. A.

48. R. L. HIPPISLEY, (under the direction of GEORGE GREENHILL) "Tables of elliptic functions" in E. P. ADAMS, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, Washington, D. C., 1922. The following errors, corrected in the first reprint of this work in 1939, were discovered in 1938 (see UMT **28**) during the checking of the table:

θ	*for*	*read*
20°	C(21) = 1.02753 28991	1.02753 29001
30°	C(36) = 1.04880 62525	1.04880 62625
55°	A(28) = 0.46253 60691	0.46252 60691
65°	A(19) = 0.31260 00376	0.31262 00376
75°	A(2) = 0.03129 20711	0.03129 20691
81°	A(14) = 0.20082 72392	0.20082 72389

G. W. SPENCELEY

Miami University
Oxford, Ohio

My attention has been called to an error in the *Smithsonian . . . Tables of Elliptic Functions*, 1939. On p. 273 in the next to the last column and the next to the last line

for 0.85562 25878, *read* 0.86562 25878.

The correct value is given on the preceding page.

E. P. ADAMS

Princeton University

49. C. JORDAN, *Calculus of Finite Differences*, Budapest, 1939.

On p. 435 are given the zeros of the Legendre polynomials $P_n(x)$ for $n = 2(1)5$. Apart from 3 unit errors in the 8th decimal place, in $n = 5$, x_0 (and $-x_4$) *for* $-$ .90617994, *read* $-$ .90617985; x_1 (and $-x_3$) *for* $-$ .53846922, *read* $-$ .53846931.

Also in the weight factors given on p. 518, apart from 2 unit errors in the 7th decimal place, A_{30} (and A_{33}), *for* .3478558, *read* .3478548.

H. E. SALZER

NYMTP

50. M. MUSKAT, F. MORGAN & M. W. MERES, *J. Applied Physics*, v. 11, 1940, p. 212. See RMT **113**.

In *MTAC*, p. 109 (Corrigenda, p. 204) reference was made to certain tables of A. KALÄHNE, 1906, reprinted in JAHNKE & EMDE 1_1–1_8, and criticized by MUSKAT, MORGAN & MERES. They claimed that the roots x_1, x_2, x_3 of the equation $J_1(x)N_1(kx) - J_1(kx)N_1(x) = 0$ when $k = 2$ would be more correct if 3.1917, 6.3116, 9.4446 were substituted for those given by Kalähne; they claimed also that they calculated the solutions of the equation for other values of k, for the purpose of problems in which they were interested. Of their revised values the first two are certainly wrong, and although Kalähne's value for x_1, $k = 2$, may be improved to read 3.1966, and the value of x_2 may be incorrectly rounded off being very nearly 6.31235, his values are the more reliable; it did not seem worthwhile to test x_3. The extra values given by Muskat, Morgan & Meres, which we have not tested are for $k = 3$, 6 zeros to 4D; $k = 5$, 7 zeros, to 5 or 4D; $k = 10$, 8 zeros to 6 or 5D. The remaining values agree precisely with those given by JAHNKE & EMDE, both being acknowledged as from the same source, Kalähne.

The writer has six or more decimal values for many of the roots given by Kalähne.

J. C. P. MILLER

51. NYMTP, *Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments*. New York, 1943. Compare RMT **151**.

C. R. COSENS of the Cambridge University Engineering Laboratory has pointed out that numerous labels of the θ curves in the contour charts on p. xv and xvii are incorrect. On p. xv starting with the 90° ray and proceeding to the right, the correct labels are as follows: 0°, −20°, −45°, −90°, *−135°*, *−160°*, *∓180°*, *160°*, *135°*, *90°*, 45°, 20°, 0°, −20°, −45°, −90° and *−135°*.

The labels on the curves in the second quadrant, which are the "reflections" of the curves of the first quadrant in the imaginary axis (90° ray) are the negatives of the labels on the corresponding curves in the first quadrant.

The labels on the curves in the third and fourth quadrants, which are the respective "reflections" of the curves of the second and first quadrants with regard to the real axis (0° and 180° rays) are the negatives of the corresponding curves in the first two quadrants.

On p. xvii starting with the imaginary axis and proceeding to the right, the correct labels are as follows: 90°, 45°, 20°, 0°, −20°, −45°, −90°, *−135°*, *−160°*, *∓180°*, *160°*, *135°*, *90°*, 45°, 20°, 0°, −20°, and −45°. Starting with the imaginary axis and proceeding to the left, the correct labels are as follows: *90°*, *135°*, *160°*, *±180°*, *−160°*, *−135°*, *−90°*, −45°, −20°, 0°, 20°, 45°, 90°, *135°*, *160°*, *±180°*, *−160°* and *−135°*. The labels on the curves in the third and fourth quadrants, which are the respective "reflections" of the curves of the second and first quadrants in the real axis are the negatives of the labels on the corresponding curves in the first two quadrants.

The above italicized numbers refer to curves incorrectly labeled in the volume.

A. N. LOWAN

52. NYMTP, *Tables of Probability Functions*, v. 1, New York, 1941. Compare RMT **91**, p. 48–51.

P. 131: The argument .6496 is missing and the argument .6497 is repeated twice.

P. 216: The first digit following the decimal point in the value of $(2/\pi^{\frac{1}{2}})e^{-x^2}$ corresponding to the argument 1.742 should read 0 instead of 5.

A. N. LOWAN

53. R. M. PAGE, *14000 Gear Ratios. . . .* New York, The Industrial Press, 1942. See RMT **87**, p. 21f.

This table has been completely checked (at my suggestion) by Mr. SIDNEY JOHNSTON, a Manchester chartered accountant, who is a great enthusiast in all forms of table work. He has, in effect, recomputed the table, and, from his known thoroughness, it is not likely that any more errors remain to be found.

The errors in Table I are all casual, being either errors of transcription from the calculating machine, or typing errors. The errors in 51/46 and 74/83, which are the only ones (apart from the obvious typing slip in 66/15) that could affect Table 3, do so. Besides these

Table 1

Page	Fraction	*For*	*Read*
37	66/15	4·000 ...	4·400 ...
64	80/42	... 915	... 905
67	111/45	111/46	111/45
68	51/46	... 659 652	... 695 652
	81/46	... 562	... 565
91	54/69	... 659 65	... 695 65
105	59/83	... 57	... 49
	74/83	·891 506 ...	·891 566 ...
	81/83	... 16	... 46
113	113/91	... 212	... 242
119	21/97	... 846 36	... 845 36
137	5/115	·143 ...	·043 ...

Table 2

Page	Col.	*For*	*Read*	C − P
145	log	589 1228	589 2228	+ 1000
146	log	682 8919	682 9819	+ 900
148	log	933 0523	933 0532	+ 9
149	N	·093 340	·094 340	...
150	log	040 3286	040 4287	+ 1001
151	log	105 3423	105 3432	+ 9
156	log	336 4119	336 4220	+ 101
157	log	364 6981	364 6991	+ 10
	log	379 5556	379 4566	− 990
	log	382 2168	383 2168	+ 10000
158	log	423 3436	423 3446	+ 10
	log	423 7471	423 7473	+ 2
160	log	463 4515	463 4416	− 99
163	log	541 3722	541 3622	− 100
164	log	561 7868	561 7968	+ 100
166	log	587 8194	587 8196	+ 2
169	log	646 4408	646 4410	+ 2
170	log	660 8023	660 8123	+ 100
174	log	727 5252	727 5255	+ 3
178	log	768 8608	768 8612	+ 4
179	ratio	37/57	35/57	...
180	log	797 8943	797 8953	+ 10
181	log	807 3085	807 3095	+ 10
184	log	836 4977	836 4979	+ 2
	log	841 5375	841 6375	+ 1000
186	log	861 1838	861 1849	+ 11
188	log	877 1541	877 1543	+ 2
190	log	897 0710	897 0720	+ 10
194	log	932 2472	932 2482	+ 10
	log	933 0528	933 0532	+ 4
196	N	·89 151	·89 157	...
	log	955 9333	955 9324	− 9
202	N	1·00 910	1·00 909	...
203	log	009 8706	009 8708	+ 2
205	log	024 3595	024 3593	− 2
	N	1·0620	1·0619	...
208	log	047 7827	047 7728	− 99
213	N	1·2181	1·2182	...
215	log	108 3384	108 3395	+ 11
	log	111 8956	110 8957	− 9999
217	log	123 0617	123 0627	+ 10
218	log	129 9695	129 9595	− 100
226	N	1·7013	1·7015	...
228	log	249 8874	249 8775	− 99
237	log	405 3320	405 3220	− 100
239	N	2·6774	2·7674	...
	log	455 9318	455 9320	+ 2
247	log	661 8978	661 8987	+ 9
	log	682 3650	683 3649	+ 9999
	log	682 5952	683 5951	+ 9999
251	log	872 8282	872 8284	+ 2
254	log	072 1070	073 1071	+ 10001
257	N	27/750	27·750	...
258	log	587 3377	587 3367	− 10

12 errors there are a number of unimportant errors of a unit in the last decimal. 30 of these are casual, but the following are systematic. On page 125 we find 35 values too small, including 100/103, so it is evident that too small a reciprocal of 103 was used. On page 137, for denominator 115, the values for numerators 9(23)101, which differ only in the first decimal, are in error. Similarly on page 140, for denominator 118, with the values for numerators 8 and 67 (=8 + 59) and 12 and 71 (=12 + 59). On page 141, 8 values are too large and 12 too small; this is mysterious, and suggests that this page was done by a different process or by a different person.

The logarithms in Table 2 would naturally be formed by subtracting the logarithm of the denominator from that of the numerator. Between 20 and 25 per cent of the logs are in error by a unit, since the logarithms of both numerator and denominator were obviously rounded off to 7 decimals before subtraction. This would lead to a theoretical expectation of one error of $+1$ unit and one error of -1 unit in every 8 values, provided no systematic effects intrude. The denominators 2, 4, 5 and 8 are very frequent, and lead to values of N that could be looked up in an ordinary 7-figure table. The fact that log 71/8 = log 8·875 (for instance) is in error seems to indicate that this process has not been adopted. On the other hand, the 7-figure logs of 2, 4 and 5 are in error by less than 0·1 of the 7th decimal, while that of log 8 only just exceeds this limit; hence, taking into account the frequency of these numbers (and of their multiples by ten) whether as numerators or denominators, the theoretical expectation above becomes less than 25 per cent. Another effect tending to reduce this percentage is numerators or denominators of 1, 10 or 100, whose logs are exact. The effect of a 5 in the 8th decimal of the log is seen in the logs of fractions containing 3 or 6 in the numerator or denominator; these bear a high proportion of systematic error. All these errors of a unit are unimportant, although most of them could easily have been avoided.

Far more serious are the 45 errors greater than a unit, although nine of them are of 2 units only, one of 3 units and two of 4; in 11 out of these 12 cases, Page's values are too low.

Table 3

Page	Fraction	*For*	*Read*
279	15/76	·196 368	·197 368
283	51/46	1·108 660	1·108 696
296	78/37	2·101 108	2·108 108
321	81/62	1·306 581	1·306 452
332	74/83	·891 506	·891 566
341	103/68	1·514 786	1·514 706
348	93/89	1·044 444	1·044 944

Table 4

Page	N	*For*	*Read*
379	1702	27 × 63	Delete
	1755	15 × 115	15 × 117
380	1936	Insert	22 × 88
	1984	Insert	32 × 62
391	4645	49 × 95	Delete
	4851	41 × 99	49 × 99
398	7772	76 × 116	67 × 116
	7790	Insert	82 × 95
400	9401	79 × 109	79 × 119

This leaves 33 errors that could vitiate a 6-figure calculation. In the list of errors, the last column shows the magnitude of each error. Those on pages 146 and 151 appear to be due to transpositions, but the remainder are subtraction errors, each of a unit in some particular column. Of the 14 errors lying between 10 ± 1, Page is too low in 12 cases; the larger errors are reasonably balanced. These errors are of a kind that anyone working by hand is liable to make, but a more experienced table-maker would have detected and eliminated them by summation or other checks. In this case, since the logs are symmetrical about log 1, they could (after typing) have been added in convenient symmetrical groups of 10 to 20, to produce totals whose last seven digits should always be 0. Alternatively, many values could have been produced in at least two independent ways; thus log 25/36 = log 50/72.

Table 3 consists of a rearrangement and rounding off of some of the values in Table 1. Apart from a very few unimportant rounding-off errors of a unit in the last decimal, there are seven errors, two of which are inherited from Table 1. The one on page 321 has been caused by combining the first three decimals of 81/62 with the second three decimals of

82/62; that this has been allowed to pass indicates that this table was not compared, *after typing*, with Table 1, as it shoud have been.

In Table 4 there are three omissions, and two cases in which the same factors have been assigned to two numbers, namely $27 \times 63 = 1701$ (not 1702) and $49 \times 95 = 4655$ (not 4645).

Summarizing, we take the view that the standard of table-making shown in this volume is not high enough to meet modern requirements. In other words, the author has not been fair to his users. He, and his publishers, and the engineering public, should be grateful to Mr. Johnston for his complete duplication of the numerical work.

L. J. C.

UNPUBLISHED MATHEMATICAL TABLES

References have been made to Unpublished Mathematical Tables in no. 7 of *MTAC*, part II, Bibliography under: AIREY, BAASMTC 2, 3, 4, 5, 6, 7, 8 (BICKLEY, Mrs. CASHEN, COMRIE, GWYTHER, HARTLEY, JOHNSTON, JONES, J. C. P. MILLER, THOMPSON), COMRIE, CORRINGTON & MIEHLE, DARMSTADT TECHNISCHE HOCHSCHULE, H. T. DAVIS, J. FISCHER, W. FISCHER, KOHLER, MORSE & HAURVITZ, NYMTP 5, 6, 7, 8, 9, 10, 11, 12, 13, TVERITIN. See also the first article of this issue, referring to a table by MILLER & BICKLEY).

24[A].—*Table of* $n!/m!$. Ms. prepared by, and in possession of H. E. SALZER, NYMTP, 150 Nassau St., New York City.

The quantities $n!/m!$ were computed for $n = 1(1)42$, $m = \overline{n - 2(-2)1}$ if n is odd, 2 if n is even. Exact values or 20 significant figures are given. A similar larger table in L. POTIN, *Formules et Tables Numériques relatives aux Fonctions Circulaires, Hyperboliques, Elliptiques*, Paris, Gauthier-Villars, 1925, p. 842–849, contains $n!/m!$ for $n = 1, 2, \cdots 50$ and $m = (n - 1), (n - 2)$ down to either 1 or $n - 25$ (exact values). In Potin's notation, he tabulates $A_m^n \equiv m(m - 1)(m - 2) \cdots (m - n + 1)$ for m up to 50 and n up to 25.

The overlapping parts of the two tables were compared and the following errors are to be noted in Potin's values: In A_{44}^{24}, the sixth from the right group of three figures, *for* 403, *read* 463. A_{38}^4, *for* 303100, *read* 303600. The following obvious errors occur in the text on p. 841: "pour m et n variant de 1 à 50" is incorrectly stated, since the n goes only as far as 25. The formula $A_m^n = A_m^{n-1}(m + n - 1)$ should read $A_m^n = A_m^{n-1}(m - n + 1)$; and on the last line, A_{m+1}^n should read A_{m+i}^n.

H. E. SALZER

25[A].—*Table of the Coefficients of the Central Factorial Polynomials.* Ms. prepared by, and in possession of H. E. SALZER, NYMTP.

This table lists the quantities B_{2m}^{2n+2}, the exact values of the coefficients of x^{2m} in the polynomials $x^2(x^2 + 1^2)(x^2 + 2^2) \cdots (x^2 + n^2)$, of degree $2n + 2$, for $n = 1(1)20$, $2m = \overline{2n + 2(-2)2}$, i.e. up to polynomials of the 42nd degree. The coefficients of x^{2m} in the central factorial polynomials $x^{[2n+2]} \equiv x^2(x^2 - 1^2)(x^2 - 2^2) \cdots (x^2 - n^2)$ are simply $(-1)^{m+n+1}B_{2m}^{2n+2}$ which are also denoted by $D^{2m}0^{[2n+2]}/(2m)!$ where $D^{2m}0^{[2n+2]}$ are usually known as the "central derivatives of zero." The values of B_{2m}^{2n+2} were obtained from the recurrence formula $B_{2m}^{2n+2} = n^2B_{2m}^{2n} + B_{2m-2}^{2n}$, starting with $B_2^2 = 1$, $B_m^2 = 0$ for $m \neq 2$, and all values on the final manuscript were checked by the relation $\sum_{m=1}^{n+1} (-1)^{m+n+1}B_{2m}^{2n+2} = 0$.

The quantities B_{2m}^{2n+2} play an important role in the calculus of finite differences whenever central factorial polynomials are to be expressed in power series. They are used to calculate

derivatives from central differences, as well as in numerical integration with central differences. Other uses are in interpolation using central differences and the calculation of generalized Bernoulli numbers. The numbers $D^{2m}0^{[2n+2]}/(2m)!$ have been scarcely tabulated before, there being a few values, up to $n = 4$, given in J. F. STEFFENSEN, *Interpolation*, p. 59 and as far as $n = 5$ in H. T. DAVIS, *Tables of the Higher Mathematical Functions*, v. 2, p. 215.

H. E. SALZER

26[A].—*Table of Coefficients for expressing Powers in Terms of Factorials.* Ms. prepared by, and in possession of H. E. SALZER, NYMTP.

This table gives the quantities C_n^m, the coefficients of $x^{(m)} \equiv x(x-1)(x-2)\cdots(x-m+1)$ in the expansion $x^n = \sum_{m=1}^{n} C_n^m x(x-1)(x-2)\cdots(x-m+1)$. The table gives the exact values of all coefficients through $m = 20$. The numbers C_n^m are usually denoted by $\Delta^m 0^n/m!$, i.e. $\Delta^m x^n/m!$ when $x = 0$, where $\Delta^m 0^n$ are commonly called "differences of zero." In his *Calculus of Finite Differences*, C. JORDAN calls the quantities C_n^m, "Stirling numbers of the second kind" and denotes them by a Gothic S in place of the C. He gives a very complete discussion of the quantities, p. 168–199, and a small table, up to $n = 12$, on p. 170. H. T. DAVIS gives a similar table in his *Tables of the Higher Mathematical Functions*, v. 2, p. 212, and in Steffensen's *Interpolation*, p. 55, there is a table up to $n = 10$. All values in these tables were found to be correct. The present table was calculated from the recurrence formula $C_{n+1}^m = C_n^{m-1} + mC_n^m$, starting with $C_n^1 = 1$ and $C_n^m = 0$ for $m > n$. All values were checked by the formula in Jordan, p. 188, equation (15) which is $\sum_{m=1}^{n+1} C_{n+1}^m = \sum_{m=1}^{n} (1+m)C_n^m$.

The quantities C_n^m occur in many important problems of finite differences, probability and number theory. They are used in converting power series into factorial series, summation of series, calculation of Bernoulli numbers, inversion of sums and series, expression of differences in terms of derivatives, expression of series of reciprocal factorials as power series, in the study of the operator $x\frac{d}{dx}$, formulation of power moments in terms of factorial moments, and in the theory of interpolation.

H. E. SALZER

EDITORIAL NOTE: The first table of C_n^m, up to $n = 9$, was given by JAMES STIRLING (1692–1770) in his *Methodus Differentialis* . . ., London, 1730, p. 8. Unfortunately it was not until after Mr. Salzer's report was in type that we observed that the table in question was not "unpublished," but was given by CAYLEY over 60 years ago, Camb. Phil. So., *Trans.*, Vol. 13, 1883, p. 2; *Coll. Math. Papers*, v. 11, 1896, p. 145–146. See also *MTAC*, p. 318, T. XXII, up to $n = 25$.

27[L].—FRANKLIN VICTOR RENO (1911–), *Tables of the Legendrean Elliptic Integral of the First Kind and the Sine-Amplitude Function.* Ms. in possession of the author, Ballistic Research Laboratory, Aberdeen Proving Ground, Maryland.

The two quantities here tabulated are

$$u = F(\phi, k^2) = \int_0^{\phi} d\phi/[1 - k^2 \sin^2 \phi]^{\frac{1}{2}},$$

and its inverse, $sn(u, k^2)$. The values of u are given to 4D for $k^2 = 0(.01)1$, and $\phi = 0(1°)90°$. The values of its inverse are tabulated for $k^2 = 0(.01)1$, and $s = 0(.01)1$, where $s = u/k$.

The tables were computed to a considerable number of decimal places beyond those tabulated in order to insure that these latter would be correct to, and including the last figure set down. Linear interpolation will yield a result correct to within one digit in the fourth decimal place.

F. V. RENO

EDITORIAL NOTE: In the tables of N. SAMOĬLOVA-ĬAKHONTOVA, RMT 50, p. 6, is a table of u for $\phi = 0(1°)\ 90°$, and $k^2 = [0(.01)1;\ 5D]$, Δ. The tables of L. M. MILNE-THOMSON, *Die elliptischen Funktionen von Jacobi*, Berlin, Springer, 1931, include a table of $sn(u, k^2)$, for $k^2 = 0(.1)1$ and $u = 0(.01)3$, with Δ; also for $k^2 = 1$, and $u = 3(.1)6.5$.

28[L].—GEORGE WELLINGTON SPENCELEY (1886–). *Tables of the seven elliptic functions A, D, (Jacobi Theta Functions) F, E, sn, cn, dn*, prepared during the years 1938–43, with the seasonal assistance of N.Y.A. students, Miami University, Oxford, Ohio.

These tables were inspired by, and follow precisely, the pattern designed by GREENHILL, and carried out by R. L. HIPPISLEY (*Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, Washington, D. C., first reprint, 1939). Hippisley's tables, to 10D, consist essentially of the four elliptic functions A, D, F, E, computed for modular angle $\theta = 5°(5°)80°(1°)89°$. His F column is the traditional value; his E column is the "periodic part of $E(\varphi)$"; his A, D columns are "normalized Jacobi Theta Functions."

The author's tables are more complete in that they are computed to 15D for modular angle $\theta = 1°(1°)89°$, and also include the sn, cn, dn functions. In these tables the E column has its traditional value. The *ms* shows 15D for all columns, with an error not greater, it is believed, than $\pm 2 \times 10^{-15}$. The original intention of the author was to cut to 12D on publication.

G. W. SPENCELEY

MECHANICAL AIDS TO COMPUTATION

11[H, Z].—D. P. ADAMS, "The quintic 'hypernom' for the equation $x^5 + Ax^3 + Bx^2 + Cx + D = 0$, a graphical method for finding the roots of polynomial equations through the fifth degree," *J. Math. Phys.*, M.I.T., v. 22, 1943, p. 78–92. 17.6 × 25.5 cm.

This paper describes a chart for the approximate determination of the roots of the reduced quintic

$$x^5 + Ax^3 + Bx^2 + Cx + D = 0.$$

By direct use of the chart, the real positive roots not exceeding 3.5 may be located in case the coefficients A, B, C, D do not exceed 10 in absolute value. By the simple expedient of changing the signs of B and D, the negative roots greater than -3.5 likewise may be determined. The author's statement that root values lie between 0 and 10 appears to be an error.

The chart consists of

(1) Three vertical linear scales for A, B, and C at the side margins,
(2) A family of curves D = constant,[1]
(3) Two families of vertical lines (L lines and M lines) bearing indices ranging from 1 to 32,
(4) Two identical horizontal non-linear "root" scales at the upper and lower margins.

The chart is unusual in that an auxiliary "S" curve must be plotted on it, a curve characteristic of the particular equation under investigation (depending, in fact, on A, B, and C only). Perhaps this feature accounts for the name "hypernom." The intersections (if any) of this S curve with the curve of the D family corresponding to the constant term of our equation determine the positive roots of the equation. The plotting of the S curve (which is preferably done on tracing paper placed over the chart), is fairly simple and consists in determining a series of points by drawing parallel lines across the grid of L and M lines.

The chart is intended as an aid to Horner's or Newton's method. The paper has six illustrative problems showing the use of the chart in various cases. Well-known methods of finding the 2 or 4 complex roots, once the real roots of the quintic have been found, are discussed. In all these examples the residual cubics and quartics happen to have coefficients not exceeding 10 in absolute value and roots less than 3.5. Some treatment of the question of quickly transforming quintic equations with large roots or large coefficients in order to be

able to use the chart, would have been interesting. The question is not trivial. The forthright method of multiplying all the roots by some small factor may not prove effective. In fact when the coefficients are thus made small the S curve is nearly coincident with the D curve and the two do not have well defined intersections.

In dealing with problems in which a large number of quintic equations with more or less the same distribution of coefficients occur it is evident that a chart of this kind would eliminate a good deal of exploratory work and make the whole process quite methodical.

D. H. L.

[1] This family is essentially the same as that given by L. I. HEWES and H. L. SEWARD: *The Design of Diagrams for Engineering Formulas and the Theory of Nomography*, New York, McGraw-Hill, 1923, p. 68, in their nomogram for the complete cubic equation.

NOTES

21. COMPUTING A TABLE OF LOG $\Gamma(x)$.—Incidental to some required computation on a problem in ballistic theory it seemed necessary to prepare a table of the logarithms of the Γ function for the range $x = [1(.01)16.99; 7D]$. This table, available on punched cards and also in typed form exhibiting the first, second and third differences, was made as follows:

The values of $\log \Gamma(x)$ for $x = 1, 1.01, \cdots, 1.99$ were punched from the 7-place table of the *Smithsonian Physical Tables*, Washington, 1934, p. 64–65. The Vega 7-place table of logarithms on punched cards allowed the use of the relation

$$\log \Gamma(x) = \log \Gamma(x-1) + \log (x-1), \tag{1}$$

for the range $x = 2, 2.01, \cdots, 2.99$, to get $\log \Gamma(x)$ for $x = 2, \cdots, 2.99$. With these newly computed values the relation (1) was again used for the range $x = 3.00, 3.01, \cdots, 3.99$, giving $\log \Gamma(x)$ for this new range. This process was applied 15 times.

In the range $16 \leqq x < 17$ exactly 15 logarithms were added to the original values of $\log \Gamma(x)$, namely:

$$\log \Gamma(x) = \log (x-1) + \log (x-2) + \cdots + \log (x-15) + \log \Gamma(x-15).$$

Since the error in each term is $< \frac{1}{2}\, 10^{-7}$, the error of any entry in our table will never reach *.000 000 8*. Because of compensation of errors the table would most likely be accurate if we rounded it off to 6 places. Of course it is absolutely accurate if rounded off to 5 places.

The whole operation was so well suited to punched-card methods that it required about a day's work for two operators.

BALLISTIC RESEARCH LABORATORY

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EDITORIAL NOTE: While this table is not new, its method of computation and its listing as on I.B.M. punched cards seemed worthy of record. W. T. RUSSELL's table of $\log \Gamma(x)$ for $x = [1(.01)50.99; 7D]$ was published in *Tracts for Computers*, no. IX (1923), ed. by K. Pearson. The Smithsonian table referred to above for $x = [1(.001)2; 7D]$ was taken from the table by LEGENDRE in his *Exercices de Calcul Intégral*, v. 2, 1816, p. 85–95, where there is a table of $\log \Gamma(x)$ for $x = [1(.001)2; 12D]$, Δ^3. A reprint of this table appeared in O. SCHLÖMILCH, *Analytische Studien*, part 1, Leipzig, 1848. The range for such a table was extended for $x = [2(.1)70(1)1200; 10D]$, with second, fourth, and sixth differences, in *Tracts for Computers*, no. VIII (1922), ed. by E. S. PEARSON. In his *Tables of the Higher Mathematical Functions*, Bloomington, Ind., 1933, p. 195–255, H. T. DAVIS has the following tables of $\log \Gamma(x)$: $x = [1(.0001)1.1; 10D]$, Δ^3; $x = [1(.001)2; 12D]$, Δ^3; $x = [2(.01)11; 12D]$, δ^2, δ^4; $x = [11(.1)101; 12D]$. $\log \Gamma(x+1)$ for $x = [1(1)1200; 18D]$ is given in two places, namely: (a) C. F. DEGEN, *Tabularum ad faciliorem et breviorem Probabilitatis computationem utilium Enneas*, Copenhagen, 1824; and (b) PETERS and STEIN, *Zehnstellige Logarithmentafel*, v. 1, *Anhang*, Leipzig, 1922, p. 61–68. A table for $\log \Gamma(x+1)$ for $x = [1(1)3000; 33D]$ was given by F. J. DUARTE in his *Nouvelles*

Tables de Log n!, Geneva and Paris, 1927. From such tables it is readily checked that there are 17 unit-errors in last figures of the Wrinch table (*MTAC*, p. 138–139). Of errors in Degen, Duarte notes (*l.c.*, p. iv) that there were 29 cases of end-figure errors of more than a half unit (all doubtless correct in the Peters and Stein edition), and the following two misprints:

$$\log 1093! = 2848.446401\ 417732\ 51(7)660$$
$$\log 1180! = 3114.2(0)8341\ 583736\ 442917.$$

For the figures in parentheses 2 and 8 should be respectively substituted.

22. EARLIEST TABLES OF S AND T.—A quotation from a letter of C. W. MERRIFIELD in *MTAC*, p. 85, suggested that the first tables of S and T were in John Newton's Trigonometry of 1658, but R.C.A. had not identified them. They occur in the *Trigonometria Britanica*, in the last section, with its own title page, *Canones Logarithmorum pro sinvbvs & Tangentibvs, ad tres primos Quadrantis Gradus, & partes Graduum Millesimas*, Londini, Ex Officina Leybourniana, MDCLVIII.

Newton gives what are now called S and T functions, to 8D, for $0(0°.001)3°$. His unit is, then, 0°.001, and the (ten-added) characteristic plus first 3 decimal places are always 6.241, or 6.242 (two columns for tangents), only the 4th to 8th decimals being printed, except at head of column. First differences of S and T are also given, rising to 132 and 264 units approximately of the 8th decimal at 3° for S and T respectively. (See the right-hand column at each of the 12 openings.) Newton calls our S and T "Diff. 1ae.", and the first difference of S and T is in a column headed "2ae.", doubtless an abbreviation for "Diff. 2ae."

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23. A NEW RESULT CONCERNING A MERSENNE NUMBER.—In *Scripta Mathematica*, v. 3, 1935, p. 112–119, there is an article by R. C. ARCHIBALD on "Mersenne's numbers," $M_p = 2^p - 1$, (p prime). This article tabulates all results known in 1935 (except one) for these numbers $p = 2$ to $p = 257$. Archibald has pointed out that the one result he had overlooked was that announced by POWERS in London Math. So., *Proc.*, v. 12, Mar. 13, 1913, p. iii, that $2^{257} - 1$ has no factor $<10{,}017{,}000$. Until recently it was not known whether 6 of the 55 Mersenne numbers were prime or composite. On August 11, 1944, I completed the proof that the smallest of these 6 numbers, M_{157}, is composite. Details concerning my computations are about to appear in Nat. Acad. Sci., *Proc.* Our present knowledge of the M_p is tabulated below.

p	Character of M_p
2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127	Prime
11, 23, 29, 37, 41, 43, 47, 53, 59, 67, 71, 73, 79	Completely factored
113, 151, 179, 223, 233, 239, 251	Incompletely factored but two or more prime factors known
83, 97, 131, 163, 173, 181, 191, 197, 211	Only one prime factor known
101, 103, 109, 137, 139, 149, 157, 241, 257	Composite but no prime factor known
167, 193, 199, 227, 229	Character unknown

H. S. UHLER

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24. Optimum-Interval Punched-Card Tables.—The article of this title by Herget & Clemence, *MTAC*, p. 173–176, raises a point of some interest, namely, that in making a table that is to be interpolable linearly, it is permissible to modify the *function* itself if the values are to be entered on punched cards. It has for some time been realized that suitably modified second or higher differences considerably reduce labour in the application of Bessel's or Everett's interpolation formulae, and modified differences, particularly modified second differences, are now well established.[1] For a printed table, however, even though designed to be interpolable, it seems never to have been questioned that function values must not be altered, and rightly, since many users need only tabular values. With a punched-card table, however, where interpolation by mechanical means is the rule, the precise values used are unseen and only the interpolated result is relevant, so that modification of the function values may well be convenient.

Herget & Clemence have, however, failed to make full use of this principle. In their illustration, a table of $1/x$, they have aimed at a linear table by choosing an interval so that $\Delta'' \leqslant 4$, with the resulting effect of less than 5 units in the extra 7th, or 'guard,' decimal. This effect is *always of the same sign* throughout each interval in any table; for $1/x$, since Δ'' is positive and the interpolation coefficient negative, this means that values obtained by ordinary linear interpolation are always in excess of the true value (apart from rounding-off errors). Thus interpolated values may be improved, on the average, if the function values, already modified by Herget & Clemence, are further modified by subtraction of $\Delta''/16$, half the maximum Δ'' effect. By this further modification the error maximum is reduced by half, and errors are no longer systematically of the same sign.

Instead of having a reduced maximum error, the extra modification may be used to allow increased intervals; in fact, for $1/x$, the max. Δ'' is $2\omega^2 x^{-3} = 8$, or $x^3 = \omega^2/4$. Taking values of ω as on p. 174, we have the approximate arguments for change of interval as follows:

ω	2	3	4	5	10	20	30	40	50
Arg.	1	1.310	1.590	1.850	2.930	4.700	6.100	7.400	8.600

This results in 924 cards for the table—a substantial improvement on the 1368 suggested by Herget & Clemence.[2]

J. C. P. Miller

[1] See, for example, L. J. Comrie, in B.A.A.S., *Mathematical Tables*, v. 1, p. xi–xii, London, 1930; *Interpolation and Allied Tables*, reprinted from the *Nautical Almanac* for 1937, p. 928–929, London, 1936. In these accounts the term 'modified differences' is given a special meaning, now accepted fairly widely. This use is not the same as that of Herget & Clemence, who use the term in place of the more familiar first *divided* differences (first used by A. De Morgan, *Differential and Integral Calculus*, p. 550, London, 1842); A. C. Aitken has suggested the symbol Δ for these (see H. Freeman, *Mathematics for Actuarial Students, Part II, Finite Differences, Probabilty & Elementary Statistics*, Cambridge, 1939, p. 41.

[2] It may be remarked that comparison with the 'usual' table of 9000 entries overestimates the value of the Optimum-Interval method. A six-figure table for 1000(1)3700(10)10000 is linearly interpolable in the usual way, has intervals 1 and 10 only, and has only 4330 entries. The improvement remains striking.

25. A Root of the Equation $10 \log x = x$.—It is hard to find anything in ordinary mathematics which L. Euler did not in some way discuss. In N **20** (p. 203) we gave various results of the nineteenth century in con-

nection with "coincidental logarithms." But already in the eighteenth century Euler had solved the problem, "Quaeratur numerus praeter 10, cujus logarithmus tabularis aequetur decimali parti ipsius numeri." (EULER, *Institutiones Calculi Differentialis*, St. Petersburg, 1755, p. 566–567; *Opera Omnia*, s. 1, v. 10, Leipzig and Berlin, 1913, p. 437–438). Euler found the number to be 1.37128857, agreeing exactly, for 7D, with what we recorded.

R. C. A.

QUERIES

10. ROUNDING-OFF NOTATION.—In RMT **157** (p. 192) attention is drawn by H.T.D. to a special use of the sign + by A. N. LOWAN and J. LADERMAN in their "Table of Fourier coefficients." In this paper the authors state that "A plus sign after an entry [to 10D] indicates that the eleventh decimal place is 5 or larger." This new use of the + seems, perhaps, a little unfortunate in view of (i) the variety of means already in use for indicating whether the final digit has been, or should be, raised or not, and (ii) T. N. THIELE's Rule, mentioned below, in which the same symbol + is used with a different meaning. It seems also worth while to note that the phrase used by Lowan and Laderman gave no hint, at any rate to the present writer, that the tenth decimal had not been raised in the usual way; it is easy to verify that, as H.T.D. notes, the tenth decimal has *not* been increased, but a hint at least sufficient to raise doubt seems essential.

As a single example of other methods for distinguishing whether the final digit should be, or has been, rounded off see *Four-Place Mathematical Tables with Forced Decimals* by F. S. CAREY and S. F. GRACE, Longmans, Green, London, 1927. In these, for some of the tables "Decimals in italics are in excess of the true value, those in thick type are in defect." Ordinary type is used when no discrimination is made.

The rule of T. N. THIELE was used by H. ANDOYER in his *Nouvelles Tables Trigonométriques Fondamentales* (*Logarithmes*), 1911, (*Valeurs naturelles*), 1, [Sines], 1915; 2, [Tangents], 1916; 3, [Secants], 1918, Hermann, Paris. Andoyer describes the rule in these words: "Si le nombre formé par les décimales qui suivent la dernière est compris entre 000 . . . et 250 . . . inclus, la dernière décimale est conservée telle quelle, sans aucun signe: si ce même nombre est compris entre 250 . . . et 750 . . ., la dernière décimale est encore conservée telle quelle, mais suivie du signe +; si enfin ce même nombre est supérieur ou égal à 750 . . ., la dernière décimale est forcée d'une unité, sans aucun signe." J. R. AIREY uses a colon ':' instead of '+' to give the same information, see, for instance, his table of the Lommel-Weber functions $\Omega_0(x)$ and $\Omega_1(x)$ in B.A.A.S., *Report*, 1924, p. 280.

In RMT **157** H.T.D. also refers to the use of 'high' and 'low' dots by L. M. MILNE-THOMSON & L. J. COMRIE in their *Standard Four-Figure Mathematical Tables* (London, Macmillan, 1931); the present writer finds this device both useful and practical.

What other devices have been used to give an accuracy beyond that which can be obtained by a bare final digit? What useful purpose is served by such of those devices as require extra printing space which could not be as well or better served by giving an extra figure, possibly correct only within 1 or 2 units, instead of the more usual $\frac{1}{2}$ unit (or sometimes 0.52 or 0.6 unit)?

J. C. P. MILLER

QUERIES—REPLIES

9. LOG LOG TABLES (Q4, p. 131).—Several years ago L. J. COMRIE had occasion to prepare a number of hectograph copies of a log log table, 4 pages (on 4 sheets), 20.5 × 33.1 cm. This is a 4-place table for the numbers 1000(10)2000 and $10^3(10^2)10^4(10^3)10^5(10^4)10^6(10^5)10^7(10^6)10^8(10^7)10^9(10^8)10^{10}$, Δ.

Another table involving log log N is that of Count ANTONIO DI PRAMPERO, in his *Saggio di Tavole dei Logaritmi Quadratici*, Udine, Tipografia G. B. Doretti e Soci, 1885, ix + 2–55 p. An account of the contents of this rare pamphlet, of which there is a copy in the Library of Brown University, was given by J. W. L. GLAISHER in his article on "Table, Mathematical," in the 11th ed. of the *Encyclopaedia Britannica*.

R. C. A.

10. ROOTS OF THE EQUATION $\tan x = cx$ (Q8, p. 203).—1. In a paper on "Vibration of power lines in a steady wind," by R. RUEDY, in *Canadian J. Research*, v. 16A, 1938, p. 147, solutions are given for two special cases of this equation $\tan x = cx$. (a) when $c = 1.35$ four roots are given as .87, 4.55, 7.76, 10.92; (b) when $c = 1.855$ the zeros are given as 1.12, 4.6, 7.785. 2. In R. GRAMMEL, "Drillungs- und Dehnungsschwingungen umlaufender Scheiben," *Ingenieur Archiv*, v. 6, 1933, p. 262, the following roots are given for the equation when $c = -1.2$: $x = 1.97, 4.88$.

H. B.

11. TABLES OF $N^{3/2}$ (Q5, p. 131; QR8, p. 204).—Further contributions to the bibliography of these tables are the following:

A. H. W. KING, *Handbook of Hydraulics*, third ed., New York, McGraw-Hill, 1939, p. 103–112; N = [0(.001)1.5(.01)21.49, 21(.1)120.9, 120(1)619; mostly 4D or 4S].

R. PEELE & J. A. CHURCH, *Mining Engineers Handbook*, third ed., New York, Wiley, v. 2, 1941, section 45, p. 26–42; $64N$=[1(1)64(8)640(64)64000; almost all to 5S]. An exactly similar table is contained in O. W. ESCHBACH, *Handbook of Engineering Fundamentals*, New York, Wiley, 1936, p. 12–28.

A. N. LOWAN

B. W. KENT & R. T. KENT, *Kent's Mechanical Engineers Handbook*, tenth ed., New York, Wiley, 1923, p. 52; N = [1(1)1000; 5S].

A. RUSSELL, *A Treatise on the Theory of Alternating Currents*, second ed., Cambridge, University Press, 1914, p. 61; $N/1000$ = [1(1)25; 4S].

A. FLETCHER
J. C. P. MILLER

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CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
Professor DAVIS, *chairman*, Professor ELDER
Professor KETCHUM, Professor LOWAN

F. Theory of Numbers
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G. Higher Algebra
Professor LEHMER

H. Numerical Solution of Equations
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I. Finite Differences. Interpolation
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EDITORIAL AND OTHER NOTICES

The addresses of all contributors to each issue of *MTAC* are given in that issue, those of the Committee being on cover 2. The use of initials only indicates a member of the Executive Committee.

Since subscriptions to *MTAC* are *payable in advance*, bills for 1945 will be sent out about November 1, 1944. Delay in payments will cause corresponding delay in mailing Number 9.

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